Chapter 7: Neutron Slowing Down – Part II

So, now that we saw all that, it's finally time to move on to the slowing-down equation. We can recall the steady-state transport equation for a homogeneous and infinite system (no time, space nor angle dependence, only energy):

$$\Sigma_t(E)\phi(E) = \int_0^\infty dE'\Sigma_s(E' \to E)\phi(E') + S(E)$$

We get an integral equation for the only variable left, i.e. the energy. And… We made it! Indeed, the slowing-down equation is simply the reduction of the Boltzmann equation to the case involving only one variable, the neutron velocity v -- or a related variable, the kinetic energy E or the lethargy u -- taken in the slowing-down domain.
We define two quantities, the arrival density $\rho(E)$ and the slowing-down current $q(E)$.

5.1. The arrival density

ρdE is the number of neutrons which, for unit time and volume, arrive in the energy interval between E and $E + dE$.

$$\rho(E) = \int_E^\infty dE'\Sigma_s(E').P(E' \to E).\phi(E')$$

With $P(E' \to E)$dE denoting the probability that a neutron scattered at the energy E' will be transferred in the energy dE between E and $E + dE$.
We often set:

$$\Sigma_s(E' \to E) = \Sigma_s(u').P(u' \to u)$$

Thus, we write:

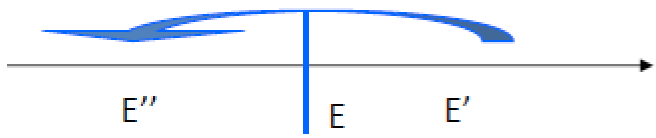
$$\rho(E) = \int_E^\infty dE'\Sigma_s(E' \to E).\phi(E')$$

We will often consider the lethargy u as a variable, and consequently write the arrival density as:

$$\rho(u) = \int_{-\infty}^{u} \Sigma_s(u' \to u)\phi(u')du'$$

5.2. The slowing-down current

$q(E)$ is the number of neutrons which, for unit time and volume, are scattered from an energy $E' > E$ to an energy $E'' < E$



$$q(E) = \int_E^{\infty} dE' \int_0^E dE'' \Sigma_s(E' \to E'')\phi(E')$$

We will once again often use the concept of lethargy:

$$q(u) = \int_{-\infty}^{u} \int_{u}^{\infty} \Sigma_s(u' \to u'')\phi(u').du'.du''$$

5.3. The first form of the slowing-down equation

The first form of the slowing-down equation involves the "arrival density".

$$\rho(u) = \int_{-\infty}^{u} \Sigma_s(u' \to u)\phi(u')du'$$

To these neutrons arriving at lethargy u (within du) after a scattering event, we must add the neutrons created directly at the lethargy u (within du) of interest, by emission from the source. Let's write this second density S.

In the problem at hand, the neutrons arriving or created at the lethargy u (to within du) only have one possible outcome, a collision (whether it causes absorption or scattering). Therefore, the sum of the two densities (arrival density and emission density) is equal to the total collision rate $\Sigma_t \phi du$, where t stands for total.

$$Sdu + \rho du = \Sigma_t \phi du$$

Therefore, one can write:

$$S(u) + \rho(u) = S(u) + \int_{-\infty}^{u} \Sigma_s(u' \to u)\phi(u')du' = \Sigma(u)\phi(u)$$

Unless you're lucky, this type of integral equation can only be solved numerically. The first form is the one used in most calculation codes.

5.4. The second form of the slowing-down equation

To characterize transfers, a second count can be performed. Instead of counting the neutrons that falls to a lethargy u (within du, still), we can count the neutrons that jump over a lethargy u. This is known as the slowing-down current, written as $q(u)$ and defined above.
Let us imagine

The slowing-down current obeys the equation:

$$q(u+du)-q(u)=\frac{dq(u)}{u}du=S(u)du-\Sigma_a(u)\phi(u)du$$

We can simplify by du :

$$\frac{dq(u)}{du}=S(u)-\Sigma_a(u)\phi(u)$$

Or, with respect to the energy:

$$\frac{dq}{dE}=S(E)-\Sigma_a\phi(E)$$

Combined with the equation given in the definition of q, this equation constitutes the second form of the slowing-down equation.

The two forms of the slowing-down equation are strictly mathematically equivalent. Indeed, if we derive the equation defining q with respect to u, we can eventually obtain the following equation:

$$\frac{dq(u)}{du}=\Sigma_s(u)\phi(u)-\rho(u)$$

By replacing this into the second form of the slowing down equation, we re-obtain the first form.

Now, let us consider elastic scattering only. In the lethargy variable, we have:
If $u-\varepsilon<u'<u$:

$$P(u'\to u)=\frac{e^{-(u-u')}}{1-\alpha}$$

$$\Sigma_s(u'\to u)=\frac{\Sigma_s(u').e^{-(u-u')}}{1-\alpha}$$

Otherwise:

$$P(u'\to u)=0$$

$$\Sigma_s(u'\to u)=0$$

Alright. There we are. We wrote the slowing-down equation, in the two forms. We explained the principle. And I'm sure everything is crystal clear for you now! A dirty, ugly, non-see-through crystal though, right?
Well, let's try and make it shiny. In order to do that, we will have to consider four different cases, of increasing difficulty.

1. Slowing-down in hydrogen without absorption
   a. $A = 1$
   b. $\Sigma_a = 0$
2. Slowing-down in hydrogen with absorption
   a. $A = 1$
   b. $\Sigma_a > 0$
3. Slowing-down in a heavier nuclei without absorption
   a. $A > 1$
   b. $\Sigma_a = 0$
4. Slowing-down in a heavier nuclei with absorption
   a. $A > 1$
   b. $\Sigma_a > 0$

5.5. Examples

5.3.1. Example 1: $A = 1$, $\Sigma_a = 0$

So, in this first example, let us consider hydrogen and no absorption. For hydrogen, we have $\alpha = 0$. Consequently, the slowing-down equation with a mono-energetic source becomes:

$$\Sigma_s(E)\phi(E) = \int_E^{\frac{E}{\alpha}} dE' \frac{\Sigma_s(E')}{(1-\alpha)E'} \phi(E') + S(E)$$

$$\Sigma_s(E)\phi(E) = \int_E^{E_0} dE' \frac{\Sigma_s(E')}{E'} \phi(E') + S_0\delta(E - E_0)$$

We define the collision density $F(E)$:

$$F(E) \stackrel{\text{def}}{=} \Sigma_s(E)\phi(E)$$

So, we obtain:

$$F(E) = \int_E^{E_0} dE' \frac{F(E')}{E'} + S_0\delta(E - E_0)$$

We try to eliminate the delta function term by separating the source neutrons from the collided flux:

$$F(E) = F_c(E) + C\delta(E - E_0)$$

neutrons with
0 collisions

The equation becomes:

$$F_c(E) + C\delta(E - E_0) = \int_E^{E_0} dE' \frac{F_c(E')}{E'} + \frac{C}{E_0} + S_0\delta(E - E_0)$$

This implies that $C = S_0$, and then:

$$F_c(E) = \int_E^{E_0} dE' \frac{F_c(E')}{E'} + \frac{S_0}{E_0}$$

neutrons with
1 collision

neutrons with
several collisions

To transform the integral equation into a differential equation, we take the derivative of the preceding equation according to:

$$\frac{d}{dx}\int_{a(x)}^{b(x)} dx'.f(x,x') = f(x,b)\frac{db}{dx} - f(x,a)\frac{da}{dx} + \int_{a(x)}^{b(x)} dx'.\frac{\partial f(x,x')}{\partial x}$$

The result is:

$$\frac{dF_c(E)}{dE} = \frac{d}{dE}\left[\int_E^{E_0} dE' \frac{F_c(E')}{E'}\right] = -\frac{F_c(E)}{E}$$

The solution is, with a constant D of integration:

$$F_c(E) = \frac{D}{E}$$

At $E = E_0$, the initial condition gives:

$$F_c(E_0) = \frac{S_0}{E_0}$$

Consequently,

$$F_c(E) = \frac{S_0}{E}$$

Eventually, we get:

$$F(E) = \frac{S_0}{E} + S_0\delta(E - E_0)$$

The solution is then:

$$\phi(E) = \frac{S_0}{\Sigma_s(E)E} + \frac{S_0}{\Sigma_s(E)}\delta(E - E_0)$$

One can notice that the flux is proportional to $1/E$. This implies that the flux diverges when $E \to 0$, which is a consequence of the hypotheses of no absorption and infinite system.

In the lethargy variable, we can write:

$$\phi(E)dE = -\phi(u)du$$

$$\phi(u) = E\phi(E)$$

Hence,

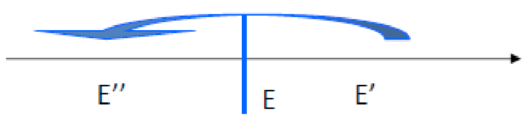
$$\phi(u) = \frac{S_0}{\Sigma_s(u)} \cong cst$$

The solution is thus constant in lethargy. This shows that the lethargy is the "right" variable to use in the slowing-down problems.

From the previous solution, we can compute the slowing-down current:

$$q(E) = \int_0^E dE'' \int_E^\infty \frac{dE'}{E'} \Sigma_s(E' \to E'')\phi(E') = S_0$$

In an infinite, non-absorbing medium, and for a given energy E, all source neutrons will sooner or later be slowed down past E.



For a distributed (in energy) source $S(E)$, we can use the previous solution as a Green function:

$$\phi(E) = \int_E^\infty dE.\frac{S(E)}{\Sigma_s(E)}\left[\frac{1}{E} + \delta(E - E_0)\right] = \frac{S(E)}{\Sigma_s(E)} + \int_E^\infty \frac{dE}{E}\frac{S(E)}{\Sigma_s(E)}$$

5.3.2. Example 2: $A = 1$, $\Sigma_a > 0$

This example assumes a medium of hydrogen, with $\Sigma_a > 0$. Actually, hydrogen does not absorb much, so this model corresponds to a mixture of hydrogen and an absorbing element of "infinite" mass, which therefore does not slow neutrons down.
In this case, the slowing-down equation becomes:

$$[\Sigma_a(E) + \Sigma_s(E)]\phi(E) = \int_E^{E_0} \frac{dE'}{E'} \Sigma_s(E')\phi(E') + S_0\delta(E - E_0)$$

We introduce again the collision density, which we decompose as the sum of an uncollided component and a collided component F_c.

$$F(E) \stackrel{\text{def}}{=} \Sigma_t(E)\phi(E) = F_c(E) + C\delta(E - E_0)$$

F_c satisfies the equation:

$$F_c(E) = \int_E^{E_0} \frac{dE'}{E'} \frac{\Sigma_s(E')}{\Sigma_t(E')} F_c(E') + \frac{\Sigma_s(E_0)}{\Sigma_t(E_0)} \frac{S_0}{E_0}$$

After differentiation, we obtain the following differential equation:

$$\frac{dF_c(E)}{dE} = -\left[\frac{\Sigma_s(E)}{E\Sigma_t(E)}\right] F_c(E)$$

The solution is:

$$F_c(E) = \frac{\Sigma_s(E_0)}{\Sigma_t(E_0)} \frac{S_0}{E} \exp\left[-\int_E^{E_0} \frac{dE'}{E'} \frac{\Sigma_a(E')}{\Sigma_t(E')}\right]$$

Probability that the first
collision at energy E_0
is a diffusion

For the flux, the solution is:

$$\phi(E) = \frac{S_0}{E\Sigma_t(E)} \frac{\Sigma_s(E_0)}{\Sigma_t(E_0)} \exp\left[-\int_E^{E_0} \frac{dE'}{E'} \frac{\Sigma_a(E')}{\Sigma_t(E')}\right]$$

We can verify that for $\Sigma_a = 0$, we fall back on the result from 4.3.1. The flux is proportional to $1/E$ out of the resonances. We can identify the resonance escape probability, i.e. the probability of being slowed down from energy E_0 to energy E, without being captured by the resonances:

$$p(E) = \exp\left[-\int_E^{E_0} \frac{dE'}{E'} \frac{\Sigma_a(E')}{\Sigma_t(E')}\right]$$

5.3.3. Example 3: $A > 1$, $\Sigma_a = 0$

For $A > 1$, we have $\alpha > 0$, and the slowing-down equation becomes:

$$\Sigma_s(E)\phi(E) = \int_E^{\frac{E}{\alpha}} \frac{dE'\Sigma_s(E')}{(1-\alpha)E'} \phi_c(E') + \frac{S_0}{(1-\alpha)E_0}$$

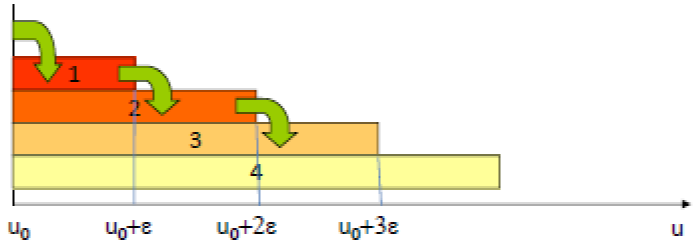
Using again the collision density F_c and after differentiation with relation to the variable E, we obtain:

$$\frac{dF_c}{dE} = \frac{1}{(1-\alpha)E}\left[F_c\left(\frac{E}{\alpha}\right) - F_c(E)\right]$$

This is not an ordinary differential equation, but rather a differential-difference equation.

Let us consider a mono-energetic source at E_0. The uncollided neutrons are at E_0, the once-collided neutrons are between E_0 and αE_0, the twice-collided between E_0 and $\alpha^2 E_0$.

In terms of lethargy, it translates to:



We need to solve separately in each interval…


Georges Placzek

For $A > 1$, the solution exhibits the Placzek transients.
$F(u)$ is discontinuous at $u = u_0 + \varepsilon$
$\frac{dF}{du}$ is discontinuous at $u = u_0 + 2\varepsilon$
$\frac{d^2F}{du^2}$ is discontinuous at $u = u_0 + 3\varepsilon$

The asymptotic solution (valid for $u > \sim u_0 + 3\varepsilon$) is given by:

$$\phi(u) \xrightarrow{u \gg u_0} \frac{S_0}{\xi \Sigma_s(u)}$$

In terms of the energy:

$$\phi(E) \xrightarrow{E \ll E_0} \frac{S_0}{\xi E \Sigma_s(E)}$$

We see that this is the hydrogen solution, divided by the slowing-down power ξ.
If we consider a mixture of different moderating isotopes, we have a similar solution:

$$\phi(E) \xrightarrow{E \ll E_0} \frac{S_0}{\bar{\xi} E \Sigma_s(E)}$$

We have there defined $\bar{\xi}$, the average slowing-down power. This is an average weighted by the corresponding scattering cross-sections:

$$\bar{\xi}(E) = \frac{\sum_i [\xi_i \Sigma_s^{(i)}(E)]}{\sum_i [\Sigma_s^{(i)}(E)]}$$

5.3.4. Example 4: $A > 1,\ \Sigma_a > 0$

The slowing-down equation with elastic and isotropic scattering is:

$$[\Sigma_a(E) + \Sigma_s(E)]\phi(E) = \int_E^{\frac{E}{\alpha}} dE' \frac{\Sigma_s(E')}{(1-\alpha)E'}\phi(E') + S(E)$$

Using the collision density $F(E)$ once again, we obtain:

$$F(E) = \int_E^{\frac{E}{\alpha}} dE' \frac{\Sigma_s(E')}{\Sigma_t(E')} \frac{F(E')}{(1-\alpha)E'} + S(E)$$

The bad news in this case is that there is no simple solution! To solve that, we'll need to wait until the next lecture on the resonant absorption.

Well, this ends the seventh lecture. If you have any question, please let me know directly or post a thread in the dedicated subreddit. Do not forget, and I can't stress this enough: if you have a question, then someone else in the class is wondering the same thing, or should be. Therefore, asking it will help you and others.

I highly recommend that you actually do the math. I did not show every single step, and it would be very beneficial for you to take over the equations and make them yours, as it helps you clear things up. It also requires some effort, but that's the price of knowledge, isn't it?

Once again, there is another thing that I should repeat. If you do not understand something, do not feel like it's your fault, and do not give up. It merely means that my explanations were not good enough. I will gladly upgrade the class by taking into account your suggestions and remarks.