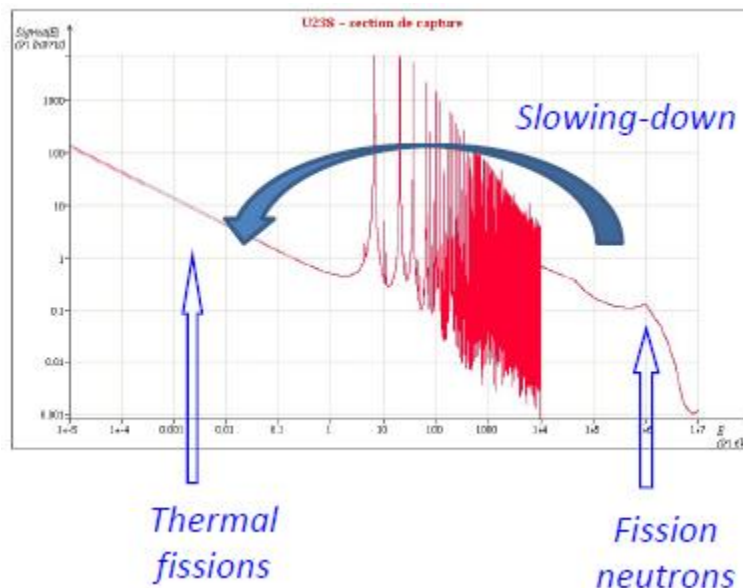


Chapter 8: The Resonant Absorption

8.1. Introduction

Yes! We did it guys! We got the neutron slowing down equation! What? Too excited? Yeah, I know, sorry about that. Especially when one considers that we are not done yet, oh boy no. Today, we are going to talk about the... Wait for it... It's not like it's in the lecture title anyway... resonant absorptions! Those resonances make it even harder to compute the flux in a nuclear reactor.



In a PWR (Pressurized water reactor for those who forgot), we saw that we needed to slow the neutrons from 2 MeV down to around 1 eV. For that to happen, neutrons must cross hundreds of resonances, several thousands barns high. If you are unsure what that means, I urge you to quickly go back to lecture #? and check it again. It will make things much clearer in the following.

In order to compute reaction rates, we need to know the spectrum (energy dependence) of the flux:

$$R_{\alpha}(\vec{r}) = \int dE \Sigma(\vec{r}, E) \phi(\vec{r}, E)$$

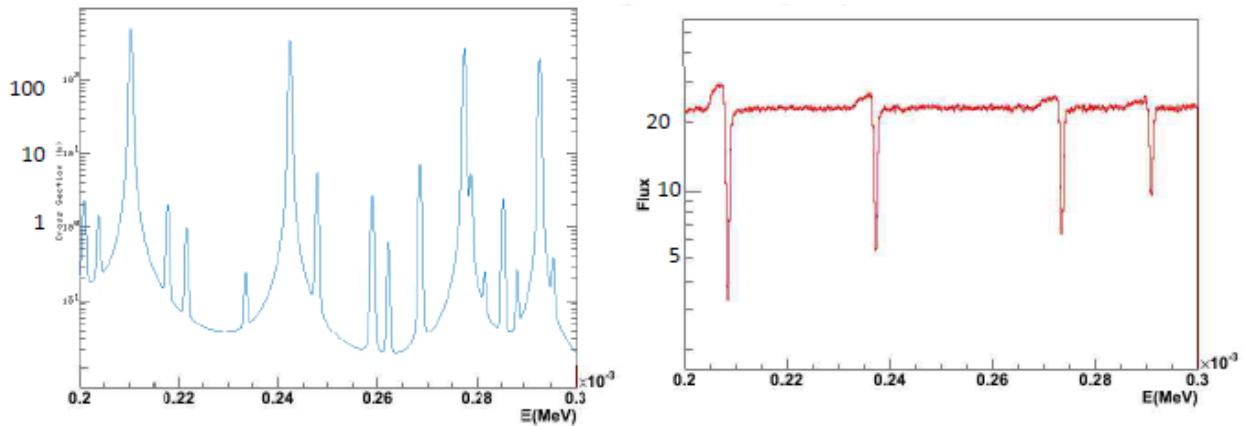
Few things to note... The stronger the variations of the cross-sections, the more accurately the flux must be known. We need reaction rates to compute multigroup cross-sections:

$$\Sigma_g(\vec{r}) = \frac{\int_{E_g}^{E_{g-1}} dE \Sigma(\vec{r}, E) \phi(\vec{r}, E)}{\int_{E_g}^{E_{g-1}} dE \phi(\vec{r}, E)}$$

The absorption rate of ^{238}U determines the resonance escape probability p , hence the k_{eff} , and the plutonium production.

8.2. Shelf-shielding

I used the CEA calculation code “APOLLO2” in order to compute a quick example. For your information, this is the simulation of a homogeneous mixture of H and ^{238}U , with a mono-energetic source at 0.31 keV.



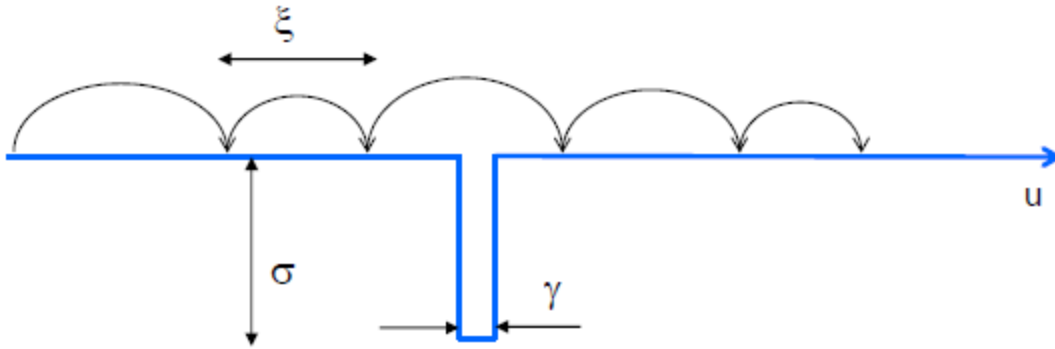
We can see that in correspondence with the peaks of the cross-section, we have dips in the flux. This is known as the self-shielding phenomenon.

The self-shielding limits the effect of absorption resonances. Indeed, the reaction rate is $\Sigma\phi$. We have an increase of Σ and a decrease of the flux ϕ , which counter-effects it. For a PWR, self-shielding divides the neutron captures of ^{238}U by a factor of nearly 14.

$$\frac{\int_{RD} dE' \Sigma_a^{U8}(E') \phi(E')}{\int_{RD} dE' \Sigma_a^{U8}(E') \int_{RD} dE' \phi(E')} \cong \frac{1}{14}$$

(RD here means “resonance domain”)

A great way to understand the resonances is through a metaphor. Let’s consider that the resonances are traps in the path (of slowing-down) of kangaroos (neutrons).



Every kangaroos that fall into the trap end up dead and absorbed by the earth. Cruel, I know. Sorry about that. Of course, kangaroos that make it are safe and sound. Well, until the next trap. The number of kangaroos falling into the trap does not depend on the depth σ of the trap (whether it's 10 meters or 2 kilometers deep does not really matter for the poor kangaroo). It only depends on the width γ of the trap compared to the length of the jumps ξ of the kangaroos.

The width of a resonance of ^{238}U is of the order of the electronVolt, while the neutrons loose on average, in each collisions with water, half of their kinetic energy. Therefore, we have for $E > 100 \text{ MeV}$:

$$\frac{\gamma}{\xi} \ll 1$$

The resonances are narrow with respect to the average energy loss per collision in the moderator, which implies that very few neutrons fall into the trap of a given resonance.

8.3. The isolated resonance

The width Γ of a resonance is the width at mid-height.

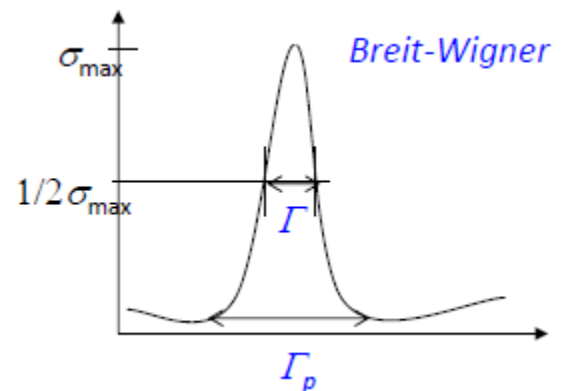
The practical width Γ_p of a resonance is the energy interval in which the resonance cross-section is bigger than the potential scattering cross-section.

It can be shown that:

$$\Gamma_p \cong \Gamma \sqrt{\frac{\sigma_{max}}{\sigma_p}}$$

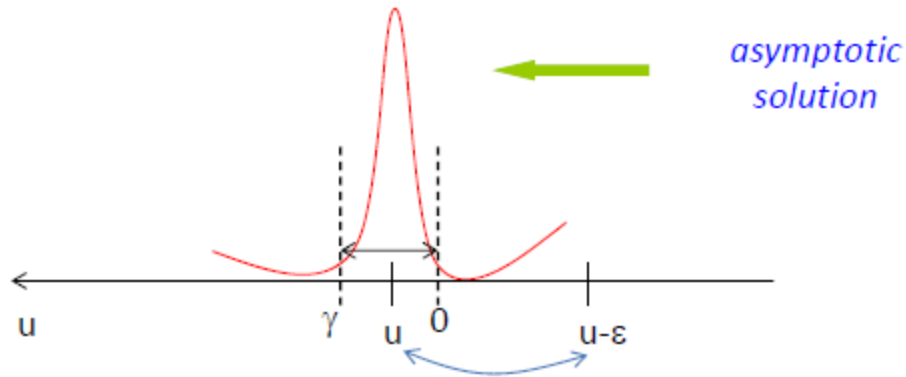
For ^{238}U we have, for some resonances:

$$\Gamma_p \cong 30\Gamma$$



8.3.1. Black resonance

Firstly, we consider an isolated black resonance, that is: all the neutrons which are scattered into the resonance will be absorbed.



Before the resonance, the flux has its asymptotic value.

$$\phi(u) \cong \frac{1}{\xi \Sigma_s}$$

We can compute the number of neutrons absorbed by the resonance using the “blackness hypothesis”:

$$1 - p_i \cong \int_0^\gamma du \rho(u)$$

p_i being the resonance escape probability, i.e. the probability of not being absorbed, and:

$$\rho(u) \cong \int_{u-\epsilon}^0 du' \Sigma_s(u') \phi(u') P(u' \rightarrow u)$$

In which $P(u' \rightarrow u) = \frac{e^{-(u-u')}}{1-\alpha}$

Consequently,

$$1 - p_i \cong \int_0^\gamma du \int_{u-\epsilon}^0 du' \frac{e^{-(u-u')}}{\xi(1-\alpha)} = \frac{1 - e^{-\gamma} - \alpha\gamma}{\xi(1-\alpha)}$$

For a very narrow resonance, we have $\gamma \rightarrow 0$, and thus:

$$1 - p_i \xrightarrow{\gamma \rightarrow 0} \frac{\gamma}{\xi}$$

8.3.2. Gray resonance

In the case of a gray resonance, the neutrons which fall into the resonance have a probability of being absorbed, $\frac{\Sigma_a}{\Sigma_t}$. You know what this means right? Some kangaroos which have fallen can be saved!

So,

$$1 - p_i \cong \int_{trap} du \frac{\Sigma_a(u)}{\Sigma_t(u)} \rho(u)$$

If, inside the resonance, we use the asymptotic value of the flux, we obtain:

$$\rho(u) = \Sigma_s(u)\phi(u) \cong \frac{1}{\xi}$$

$$1 - p_i \cong \int_{trap} \frac{du \Sigma_a(u)}{\xi \Sigma_t(u)}$$

Since the absorption of each resonance is small, we can use the limited development:

$$p_i \cong 1 - \int_{(i)} \frac{du \Sigma_a(u)}{\xi \Sigma_t(u)} \cong \exp \left[- \int_{(i)} \frac{du \Sigma_a(u)}{\xi \Sigma_t(u)} \right]$$

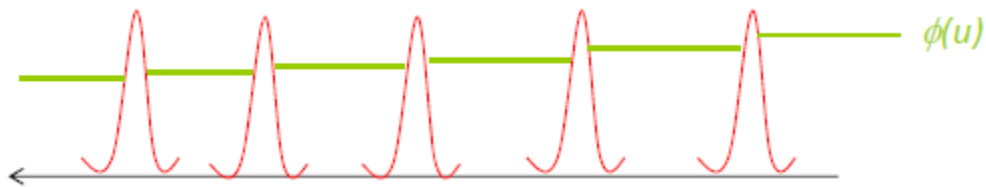
Consequently, for a succession of gray resonances, we can write:

$$p \cong \prod_i p_i \cong \exp \left[- \int \frac{du \Sigma_a(u)}{\xi \Sigma_t(u)} \right]$$

This is the resonance escape probability, or, in our case, the trapped kangaroo escape probability...

8.4. Factorization of the flux

For a sequence of well-spaced resonances, as drawn below, we can tell a few things.



Outside the resonances, the flux has its asymptotic value:

$$\phi(u) \rightarrow \frac{1}{\xi \Sigma_s}$$

At each resonance, a fraction p_i of the neutrons are absorbed. Of course, that means that inside the resonances, we expect the flux to decrease.

We can compute the slowing-down current $q(u)$ as:

$$q(u) = S_0 \prod_j^{u_j < u} p_j$$

We now introduce the factorization of the flux.

$$\phi(u) = \Phi(u)\varphi(u)$$

Φ is the asymptotic flux. It has the dimension of... a flux.

$$\Phi(u) = \frac{1}{\xi\Sigma_s}$$

φ is the self-shielding factor, sometimes known as the fine structure flux. It is dimensionless. It is equal to 1 outside the resonances, and plummets inside.

For the resonance escape probability of a single resonance, and considering the scattering cross-section Σ_s to be approximately constant, we have:

$$\begin{aligned} p_i &= \exp \left[- \int_{E_i} dE' \Sigma_a(E') \phi(E') \right] \\ &= \exp \left[- \frac{1}{\xi\Sigma_s} \int_{E_i} \frac{dE'}{E'} \Sigma_a(E') \varphi(E') \right] \end{aligned}$$

This expression shows the competition between scattering (favorable to the neutron balance) and the absorption.

8.5. Effective resonance integral

We can define the effective resonance integral as being:

$$I_{eff}^{(i)} \stackrel{\text{def}}{=} \int_{E_i} \frac{dE'}{E'} \sigma_a^A(E') \varphi(E')$$

I_{eff} has the dimensions of a cross section, measured in barns. It is defined in terms of microscopic quantities. σ_a^A is the resonant cross-section of the absorber considered.

Now, we can rewrite the resonance escape probability, factoring in the effective resonance integral.

$$\begin{aligned} p_i(E) &= \exp \left[- \frac{1}{\xi} \int_{E_i} \frac{dE'}{E'} \frac{\Sigma_a(E')}{\Sigma_s(E')} \varphi(E') \right] \\ p_i(E) &= \exp \left[- \frac{N_A}{\Sigma_s(E_i)} I_{eff}^{(i)} \right] \end{aligned}$$

N_A being the number of atoms per cm^3 for the absorber.

This formula, while being apparently simple, is also kind of physically intuitive. It expresses the fact that the resonance escape probability is the result of a competition between the numerator describing absorption in the resonance traps of the fuel (not good) and the denominator, the slowing-down by the moderator (good).

It is also possible now to write down the multigroup cross-section in terms of the effective resonance integral:

$$\Sigma_{a,g} = \frac{\int_{E_g}^{E_{g-1}} dE \Sigma_a(E) \phi(E)}{\int_{E_g}^{E_{g-1}} dE \phi(E)}$$

If we use the asymptotic form of the flux, we have:

$$\phi(E) \cong \frac{1}{\xi E \Sigma_s} \varphi(E)$$

And so:

$$\Sigma_{a,g} \cong \frac{N_A \sum_{i \in g} I_{eff}^{(i)}}{\xi \Delta u_g}$$

8.6. The self-shielding factor

Let us consider now a homogeneous mixture made of a moderator which does not absorb and has no resonances; and an absorber which absorbs and scatters.

$$\Sigma_t^M(E) \cong \Sigma_s^M(E) = \Sigma_p^M$$

$$\Sigma_t^A(E) = \Sigma_a^A(E) + \Sigma_s^A(E) = \Sigma_a^A(E) + \Sigma_{s,res}^A(E) + \Sigma_p^A$$

We consider the slowing-down in a resonance but we keep only the elastic and isotropic scattering.

$$\Sigma_t(E) \phi(E) = \int_E^{E/\alpha_M} \frac{dE' \Sigma_s^M \phi(E')}{E' (1 - \alpha_M)} + \int_E^{E/\alpha_A} \frac{dE' \Sigma_s^A \phi(E')}{E' (1 - \alpha_A)}$$

If the resonance is narrow with relation to the average energy loss in a collision, we can use the asymptotic value of the flux in the first integral:

$$\Delta E|_M = \frac{1}{2} (1 - \alpha_M) E_0 \gg \Gamma_p$$

$$\phi \cong 1/E$$

And so, in that case:

$$\int_E^{E/\alpha_M} \frac{dE' \Sigma_S^M \phi(E')}{E' (1 - \alpha_M)} \cong \frac{\Sigma_S^M}{(1 - \alpha_M)E'} \int_E^{E/\alpha_M} \frac{dE' 1}{E' E'} = \frac{\Sigma_S^M}{E}$$

We have thus two cases to consider. The narrow resonance approximation, and the wide resonance.

8.6.1. Narrow resonance approximation

In this case, the resonance is narrow with relation to the average energy loss in a collision in the absorber.

$$\Delta E|_A = \frac{1}{2}(1 - \alpha_A)E_0 \gg \Gamma_p$$

$$\phi \cong 1/E$$

$$\int_E^{E/\alpha_A} \frac{dE' \Sigma_S^A \phi(E')}{E' (1 - \alpha_A)} \cong \frac{\Sigma_p^A}{(1 - \alpha_A)E'} \int_E^{E/\alpha_A} \frac{dE' 1}{E' E'} = \frac{\Sigma_p^A}{E}$$

The slowing-down equation then becomes:

$$\Sigma_t(E)\phi(E) = \frac{\Sigma_S^M + \Sigma_p^A}{E}$$

Therefore, we have the explicit solution:

$$\phi_{NR}(E) = \frac{\Sigma_S^M + \Sigma_p^A}{E\Sigma_t(E)}$$

8.6.2. Wide resonance

In this case, the resonance is wide with relation to the average energy loss in a collision in the absorber.

$$\Delta E|_A = \frac{1}{2}(1 - \alpha_A)E_0 \ll \Gamma_p$$

$$\alpha_A \rightarrow 1$$

$$\int_E^{\frac{E}{\alpha_A}} \frac{dE' \Sigma_S^A \phi(E')}{E' (1 - \alpha_A)} \xrightarrow{\alpha_A \rightarrow 1} \Sigma_S^A(E)\phi(E)$$

The slowing-down equation becomes:

$$\Sigma_t(E)\phi(E) = \Sigma_s^A\phi(E) + \frac{\Sigma_s^M}{E}$$

We thus have the explicit solution:

$$\phi_{WR}(E) = \frac{1}{E} \frac{\Sigma_s^M}{\Sigma_t(E) - \Sigma_s^A(E)}$$

If we take the example of Uranium 238, we can write the results for the first resonances:

E_0 (eV)	σ_{max} (barn)	Γ_p (eV)	$(1 - \alpha)E_0$ (eV)
6.67	22900	1.4	0.11
20.9	32000	2.04	0.35
36.8	39900	3.82	0.62
80.74	2600	0.43	1.36
102.47	18600	4.39	1.72
145.58	600	0.2	2.45
165.27	2400	0.35	2.78

In green, the wide resonances, and in yellow the narrow ones. For higher energies, all the resonances of U238 are narrow.

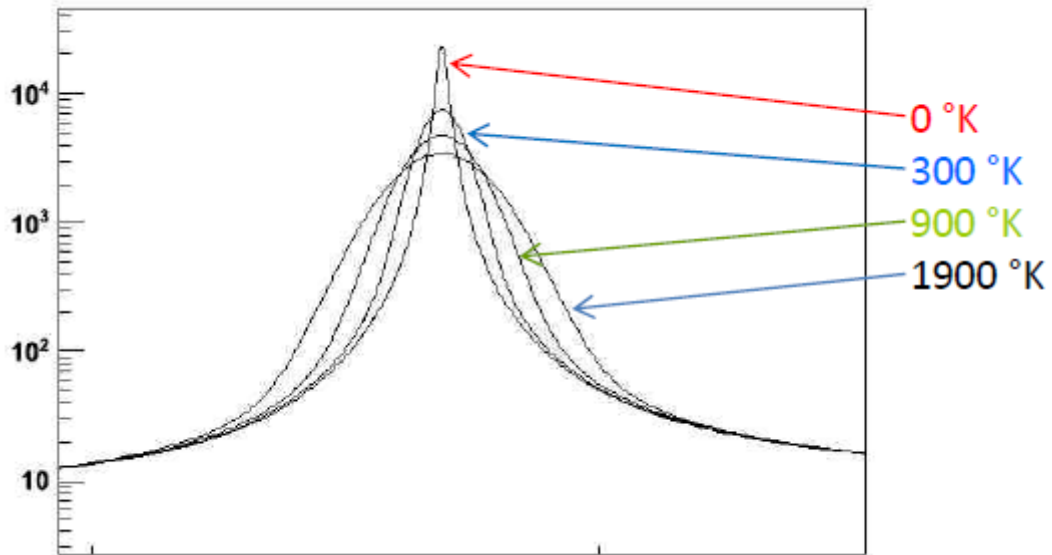
The narrow resonance and the wide resonance are called self-shielding models. Other models exist, such as the Livolant-Jeanpierre, or the sub-groups model (used for fast reactors). I won't go into those. Self-shielding models are approximations, even if they can be fairly accurate (and complicated). It is interesting to note that they are not convergent models, so it is difficult to control the error you get.

8.7. The Doppler effect

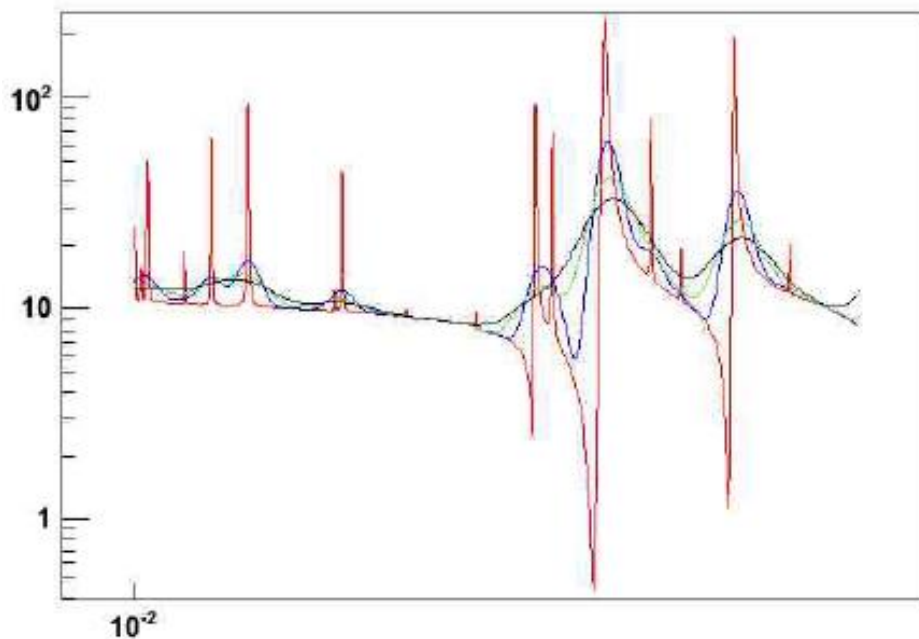
Right, so we have supposed throughout the classes that the target nuclei were at rest, therefore neglecting their thermal motion with relation to the speed of the neutrons. Indeed, cross-sections are evaluated at 0K.

This hypothesis is of course reasonable for scattering, but is no longer valid when we're dealing with resonance absorption. Ah, bugger. Indeed, the width of the resonance, roughly around 0.1 eV, is of the same order of magnitude as the thermal energy of the medium.

So what, do you say? Well, that means we need to correct the theoretical cross-sections (obtained at 0K) for the temperature T. That is done by convoluting the cross-sections at 0K with a Maxwell distribution at T.



So, by zooming in on the first resonance of Uranium 238 (at 6.67 eV), we can see the correction made. The resonances become wider but shorter. On the second graph below, you can see the effect on a whole spectrum.



So, it grows wider but shorter. As a result, the resonance integral is almost unchanged. However, the effective resonance integral increases with the square root of the temperature. Effectively, we have:

$$I_{eff}(T) \cong I_{eff}(T_0)[1 + \beta(\sqrt{T} - \sqrt{T_0})]$$

For fissile materials, temperature effects on fission and capture tend to compensate. The main effect is on the resonant captures of Uranium 238. For a PWR, the temperature effect on reactivity is around -2 to -3 pcm (10^{-5}) per degree.

To illustrate that:

If a local perturbation increases the reactivity, the power will increase. This will make the temperature increase, which in turn will make the reactivity decrease.

The Doppler effect is an example of negative feedback, and helps stabilizing the reactor.

Since the Doppler effect depends on the temperature of the fuel (Uranium 235 produces the power and Uranium 238 absorbs the neutrons), the effect is almost instantaneous (small scale effect).

8.8. Conclusion on the resonant absorptions

Remember, everything we have seen are models in an homogeneous and infinite medium... Heterogeneous configurations are obviously a lot harder to compute. In those case, we need to treat space, angle with the energy. You go from one dimension to six. More sophisticated models are needed. In the future, with the progress of computing power, Monte-Carlo method (you simulate all the neutrons stochastically) could become viable (and exact).

Exercises will come (I have to publish the problem sets for the previous lectures too) as soon as I can. Thank you for following, and don't hesitate to say if you think it is unclear. Of course, it is always useful to be able to look at the big picture, so try and look back at how this intertwines with the previous lectures. If it seems to you that it doesn't at all, that means I did a poor job, and I will try my best to correct that.

Well, this ends the eighth lecture. If you have any question, please let me know directly or post a thread in the [dedicated subreddit](#). Do not forget, and I can't stress this enough: if you have a question, then someone else in the class is wondering the same thing, or should be. Therefore, asking it will help you and others.

I highly recommend that you actually do the math. I did not show every single step, and it would be very beneficial for you to take over the equations and make them yours, as it helps you clear things up. It also requires some effort, but that's the price of knowledge, isn't it?

Once again, there is another thing that I should repeat. If you do not understand something, do not feel like it's your fault, and do not give up. It merely means that my explanations were not good enough. I will gladly upgrade the class by taking into account your suggestions and remarks.

I also wanted to apologize for the delay. My daughter was born three weeks ago, and that takes time!